Infinity – The Very Short Introductions Podcast – Episode 65

Rebecca Parker 00:07
Welcome back to The Very Short Introductions Podcast. From public health to Buddhist ethics, soft matter to classics, and art history to globalization, we'll showcase a concise and original introduction to a wide range of subjects for, wherever your curiosity may take you. So here is today's very short introduction.

Ian Stewart 00:26
My name is Ian Stewart. I'm a retired professor of Mathematics at Warwick University. And being a mathematician, I've run into infinity in some form or another, virtually every day of my mathematical life. And it's a fascinating topic. It's not just mathematicians who are interested in the infinite. Philosophers, naturally, are fascinated by the rather complicated and paradoxical nature of infinity. And theologians have debated the notion of infinity relative to various deities in various religions. I want to mostly focus on the maths.

Ian Stewart 01:06
But we run into infinity very early on. If you ask children, what's the biggest number; when they're very young, they will name the biggest number they know. So, depending on the age, they'll say, 10, or 100, or a million. But, fairly early on, they realize that however bigger number they name, there's always a bigger one, and the easy way to get a bigger one is just to add 1. So if you think of two children playing a game, see who can name the biggest number, and keep going until one of them can't. Well, basically, the game goes on forever, because very rapidly, they say, "Oh, one bigger than what you just said." And they keep doing that. So there is no biggest number.

Ian Stewart 01:51
So in some sense, the collection of all possible numbers--I'm just thinking of whole numbers here--is infinite. And we kind of get used to that, we don't think very much about it. But if you start examining it more closely, it's a very tricky notion. And in fact, there are many different kinds of infinity, as I'll get around to.

Ian Stewart 02:14
So the first recorded investigations of infinity are by ancient Greek philosophers. About 580 BC, Anaximander, wondering where everything in the universe came from, came up with something he called "apeiron," which is a kind of formless chaos, out of which everything is created. It's infinite, so you can make things from it without ever running out. It's a bit like the quantum mechanical idea that particles and antiparticles can suddenly wink into existence from a vacuum. But I don't think the ancient Greeks were really thinking about quantum mechanics at that point. So it's a slightly vague concept, but it's the first recorded one.

Ian Stewart 03:01
The first mathematical analysis of the infinite is about a hundred years later. And this is the ancient Greek Zeno; you may well have heard of Zeno's paradoxes. And Zeno came up with four different paradoxes-- paradox meaning that the logic just did not work properly, it didn't matter whether you

thought something was true or false, you ran into trouble. And the most famous one, and the one I'm going to talk about, is Achilles and the tortoise. So Achilles, who is big and hulky and very, very athletic, he's going to run a race against the tortoise. And he knows the tortoise is a lot slower. So he said, "Okay, look, I'll give you an advantage. You can start 10 meters further ahead than me, and then we'll start the race. And I'm going to beat you." And the tortoise said, "Oh, no, you won't, by the time you get 10 meters, I'll have moved on a bit. And by the time we get to where I was, then I'll have moved on a little bit more. And so you will have to do infinitely many different things before you can catch me." Of course, they run the race, and Achilles runs past the tortoise at high speed. And the tortoise is sitting there thinking, "Well, the logic seemed to work, but I've lost the race." So this is an example of what kind of assumptions you make about infinity. Is it actually possible to do infinitely many things in a finite time? And the modern answer is that in the mathematical models we use, yes, it is. Basically Achilles is right, the tortoise is wrong. But it's not always that simple.

Ian Stewart 04:46
Aristotle, who is perhaps the most famous of the Greek philosophers, came up with a very important distinction between what he called actual infinity and potential infinity. So let's go back to the whole numbers. You start counting 1, 2, 3, 4, 5. And this process can go on and on, and, in principle, it never stops, because, however far you get, you can always go a bit further. Now, that's an example of what Aristotle called potential infinity. We never actually get to infinity, we just keep going and never stop. But at any time, we've only counted finitely many numbers. But there isn't a biggest one. So potentially, the number system is infinite, but we never grasp the entire infinitude of it as a single thing. Actual infinity would be something that is of itself genuinely infinite. And actually, most uses of infinity in mathematics are potentially infinity or can be recast and reworded so that you're never actually facing actual infinities. And that was true until about 1850 when the whole game changed.

Ian Stewart 06:10
Throughout history, people, often very famous people, have come across infinity, thought about it, noted various interesting properties of it, and given their view of what's involved. And Galileo, the great Italian scientist and mathematician, wrote a book in 1638, Discourses and Mathematical Demonstrations Relating to Two New Sciences. And he had some characters who were discussing things. And at some point, they're discussing numbers, and one of them says, "Well, we all know about the numbers, 1, 2, 3, 4, 5, 6, and so on. And we know what the square of a number is; to square a number you multiply by itself, so 1 squared is 1, 2 squared is 2 times 2, that's 4, 3 squared is 3 times 3, that's 9. Sixteen, 25, 36, and so on. And a lot of numbers are not squares. So rather, obviously, there are more numbers than there are square numbers." The other one says, "Yes, but I can match them up. Every number corresponds to a square number, namely, its own square. So the first square is 1, the second square is 4, or the third square is 9, and so on. And since I can match up every number with every square, and conversely, there are exactly as many squares as there are numbers." So on the one hand, there are obviously more numbers than squares. And on the other hand, the numbers are obviously equal. And Galileo, basically, at that point, throws up his hands and says, "Well, what this shows is that talking about equality, greater or less, doesn't apply to infinity." And for a long, long time, it's left like that.

Ian Stewart 08:04

Now the other place that we encounter infinity is in arithmetic, recurring decimals. We learned about fractions, we learn about decimals, we come across 1/3 and we're told it's naught point 33333 going on forever. And that the number pi, which comes up in connection with circles, is 3.141, etc., and that goes on forever, as well. And one of them, it keeps repeating the same number and the other one doesn't, but they both go on forever. And at school, we kind of slide over all of that. The teachers don't want to get too involved in what it means. But mathematically, this is a very subtle point, that decimal numbers that go on forever are not something you can actually write down. You can write down a rule to calculate them, but you can't write them down. And that's a kind of potential infinity as well. We can always work out the first hundred decimal places or the first thousand but we can't write down the lot. Why is this important? Why should we care? Because a lot of today's society actually depends upon the mathematics of infinity, and have the kind of flipside of infinity which is the infinitely small, infinitesimals, numbers or things like numbers, which are so small, that in certain circumstances, you can almost pretend they don't exist. And this is the basis of calculus.

Ian Stewart 09:38
So calculus is used to find areas, to find volumes, to calculate the speed of objects, to calculate accelerations, and the whole of mathematics of calculus rests upon adding up infinitely many infinitely small pieces and getting a sensible answer. And so the mathematicians of the 17th, 18th, 19th centuries had to grapple with that idea. The engineers and scientists and the physicists and chemists just picked it up and ran with it and said, "We don't really care about the foundational details of this, this stuff works, let's use it." And this is why we now have satellites in orbit. It's how it's possible to land on the moon. It's how cars are designed, it's kind of how bits of the Internet work. It's how electricity works. All of these things we understand using the mathematics of calculus, which is founded upon notions of infinitely large and infinitely small. So this stuff is actually useful, it's indispensable.

Ian Stewart 10:44
A very different place where we encounter infinity is, surprisingly, in art, in paintings. Before the Renaissance period, paintings were often very beautiful, but slightly crude in the sense that things like perspective of buildings and landscapes didn't really work, that the painting didn't look like a photograph. It was a nice painting. The Renaissance painters discovered mathematical techniques for getting very, very accurate perspective. And so when they draw a landscape with buildings, and maybe it's Venice, and there's a canal, the stuff up front looks bigger, the stuff at the back looks smaller, but also the whole thing fits together so that the eye sees it almost as a photograph of a three dimensional object. And infinity comes into the mathematics of that; think about railway lines, suppose you are standing--it's bit dangerous--but you stand in the middle of a straight railway line, which is going on for some considerable distance. And basically, when you look at it, the lines seem to converge, because the further away the lines get, the closer together, they become, as far as the eye is concerned. They're actually parallel, but they seem to get closer together. And if you do this using Euclid's geometry, then you would draw parallel lines on a plane and you would stand between them and look to where they go, and they would seem to converge to a point that is infinitely far away. And it's an entire line like a horizon, which looks like the edge of the infinite plane. But an infinite plane doesn't have an edge. So there is something called the line at infinity, which is a geometric example of a concept motivated by thinking about the infinite.

Ian Stewart 12:39
And it leads to what's called projective geometry. In Euclid's geometry, two straight lines either meet or they're parallel and they don't meet. In projective geometry, two straight lines always meet; the parallel ones in Euclid's geometry meet on the line at infinity. So you're kind of completing the infinite plane with a tangible boundary, where lines can meet. And so projective geometry became very, very important, both for art and later for mathematics.

Ian Stewart 13:14
What about the real world? Is anything in the real world genuinely infinite? While not too long ago, I think most physicists--and they're the people who grapple with this mainly--would have said, mostly not. Usually, when you meet an infinity in your calculations, it means your theory is breaking down. There's a theory of the rainbow to do with how the light is focused by a raindrop. And the classical version of this predicts that the intensity of light on the rainbow should be infinite. And clearly it's not. So you need a slightly different theory there. It's a wave theory of light, and then things work out better. But one question is, how big is the universe? And basically, we have no idea. It's very, very big. There's a limit to what we can observe. And physicists do discuss whether the universe is actually infinite, and we're just seeing a finite portion of it, or whether the whole thing is finite; it might even be curled up into a sort of four-dimensional ball. So there's a possibility there for a real infinity.

Ian Stewart 14:19
The other place where you might find a real infinity, certainly in the theory, is a black hole. And you will have heard of a black hole; it's what happens when a very heavy star collapses under its own gravity. And it collapses so much and become so dense, that the attractive force of gravity stops light escaping from it; the light can't get fast enough to escape the gravitational field. Now, the theory is that inside the black hole, all of the matter that collapsed actually condenses right down to a single mathematical point of infinite density. We can't get inside a black hole so we don't know what the reality is, but at the moment, the best theories we have are predicting a real infinite density point. My suspicion is that that will eventually be overturned by a bit of theory, but that's a case where real infinities are turning up in physics.

Ian Stewart 15:15
And the final mathematical point I'd like to make was the amazing discovery by Georg Cantor, in 1874, that some infinities are bigger than others. Now, that sounds ridiculous, it's infinitely big, how can anything else be, some infinite because we even bigger than other infinitie? And his example was the counting numbers, 1, 2, 3, 4, 5, and so on, going on forever, thought of as an actually infinite set, this isn't actual infinity not potential anymore. That's a smaller infinity than the infinite decimals, all of the possible infinite decimals, .33333, recurring, pi, 0.714526, you name it, it's up to you, carry on forever. Cantor proved that there are more infinite decimals than there are counting numbers. You can't match them up like the counting numbers match Galileo's squares. So there are different sizes of infinity.

Ian Stewart 16:12
And so I'd like to leave you with the thought that infinity starts out as a sort of philosophical concept, is turned into mathematics, the mathematics is turned into physics, and then some of the mathematics

comes up with things that nobody had thought of, until a couple of 100 years ago, which also, in fact, have turned out to be very important. So infinity is beautiful, puzzling, and well worth thinking about.

Rebecca Parker 16:41
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